ON LINEAR INDEPENDENCE OF INTEGER SHIFTS OF COMPACTLY SUPPORTED DISTRIBUTIONS

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ABSTRACT. Linear independence of integer shifts of compactly supported functions plays an important role in approximation theory and wavelet analysis. In this note we provide a simple proof for two known characterizations of linear independence of integer shifts of a finite number of compactly supported distributions on $\mathbb{R}^d$.

By $l(\mathbb{Z}^d)$ we denote the space of all complex-valued sequences $v = \{v(k)\}_{k\in\mathbb{Z}^d} : \mathbb{Z}^d \to \mathbb{C}$ on $\mathbb{Z}^d$. In particular, by δ we denote the Dirac/Kronecker sequence on $\mathbb{Z}^d$ such that $\delta(0) = 1$ and $\delta(k) = 0$ for all $k \in \mathbb{Z}^d\backslash\{0\}$. Let $\phi_1, \dots, \phi_r$ be compactly supported distributions on $\mathbb{R}^d$. The integer shifts of $\phi_1, \dots, \phi_r$ are (globally) *linearly independent* if

$$\sum_{\ell=1}^{r} \sum_{k\in\mathbb{Z}^d} v_\ell(k)\phi_\ell(\cdot - k) = 0 \tag{1}$$

for some sequences $v_1, \dots, v_r \in l(\mathbb{Z}^d)$, then we must have $v_1(k) = \dots = v_r(k) = 0$ for all $k \in \mathbb{Z}^d$.

As usual, by $\mathscr{D}(\mathbb{R}^d)$ we denote the space of all compactly supported C^∞ (test) functions on $\mathbb{R}^d$. For a locally integrable function ϕ on $\mathbb{R}^d$, we shall use the following pairing:

$$\langle \phi, h\rangle = \int_{\mathbb{R}^d} \phi(x)\overline{h(x)}dx, \qquad h \in \mathscr{D}(\mathbb{R}^d),$$

where $\overline{h(x)}$ is the complex conjugate of $h(x)$. For a general distribution ϕ and $h \in \mathscr{D}(\mathbb{R}^d)$, $\langle \phi, h\rangle := \phi(\overline{h})$ and we define $\langle h, \phi\rangle := \overline{\langle \phi, h\rangle}$. For $x = (x_1, \dots, x_d)^\mathsf{T}$ and $z = (z_1, \dots, z_d)^\mathsf{T} \in \mathbb{C}^d$, we define $z \cdot x := z_1x_1 + \dots + z_dx_d$ and moreover, we define $z^x := z_1^{x_1} \cdots z_d^{x_d}$ if all $x_1, \dots, x_d$ are integers and $z_1, \dots, z_d$ are nonzero. For a compactly supported function $\phi \in L_1(\mathbb{R}^d)$, its Fourier-Laplace transform $\widehat{\phi} : \mathbb{C}^d \to \mathbb{C}$ is defined to be

$$\widehat{\phi}(z) := \int_{\mathbb{R}^d} \phi(x)e^{-iz\cdot x}dx = \langle \phi(x), \overline{e^{-iz\cdot x}}\rangle, \qquad z \in \mathbb{C}^d.$$

The Fourier-Laplace transform can be naturally extended to compactly supported distributions ϕ as $\widehat{\phi}(z) := \widehat{\phi h}(z) = \langle \phi(x), \overline{h(x)e^{-iz\cdot x}}\rangle$, where $h \in \mathscr{D}(\mathbb{R}^d)$ takes value 1 in a neighborhood of the support of ϕ. For a compactly supported distribution ϕ on $\mathbb{R}^d$, its Fourier-Laplace transform $\widehat{\phi}$ is an analytic function in $\mathbb{C}^d$.

The problem of linear independence of integer shifts of functions originated from investigation of multivariate splines. de Boor and Höllig [2] considered linear independence of integer shifts of a box spline and obtained a necessary condition, which is confirmed to be also a sufficient condition by Jia [6]. Ron [9] characterizes linear independence of integer shifts of a compactly supported distribution (that is, $r = 1$) in terms of its Fourier-Laplace transform. The general case of linear independence for any $r \in \mathbb{N}$ has been established in Jia and Micchelli [7] by studying solutions of certain systems of partial difference equations ([4, 7]). A different characterization of linear independence of integer shifts of compactly supported distributions is given as a special case of Ben-Artzi and Ron [1] with an extension by Zhao [11] in terms of compactly supported dual functionals. See Ron [10] for an

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excellent survey on many interesting results about shift-invariant spaces and linear independence. For the importance of linear independence in approximation theory, see [1, 2, 4, 6, 7, 9, 10, 11]. For the role and application of linear independence in wavelet analysis, see [3, 5, 8, 10] and references therein.

Due to the importance of the two characterizations of linear independence of integer shifts of a finite number of compactly supported functions and distributions in approximation theory and wavelet analysis, we provide a simple self-contained proof here.

Main Theorem. *Let $\phi_1,\dots,\phi_r$ be compactly supported distributions on $\mathbb{R}^d$. The following statements are equivalent:*

(i) *The integer shifts of $\phi_1,\dots,\phi_r$ are linearly independent.*

(ii) *$\{\widehat{\phi_\ell}(z+2\pi k)\}_{k\in\mathbb{Z}^d}, \ell=1,\dots,r$ are linearly independent for all $z\in\mathbb{C}^d$, that is, there do not exist $\zeta\in\mathbb{C}^d$ and $c_1,\dots,c_r\in\mathbb{C}$ such that $|c_1|+\dots+|c_r|\ne 0$ and*

$$\sum_{\ell=1}^r c_\ell\widehat{\phi_\ell}(\zeta+2\pi k)=0, \qquad \forall\, k\in\mathbb{Z}^d. \tag{2}$$

(iii) *There exist compactly supported $C^\infty(\mathbb{R}^d)$ functions $\tilde\phi_1,\dots,\tilde\phi_r\in\mathscr{D}(\mathbb{R}^d)$ such that*

$$\langle \tilde\phi_m, \phi_\ell(\cdot-k)\rangle=\delta(m-\ell)\delta(k), \qquad \forall\, k\in\mathbb{Z}^d \quad \text{and} \quad \ell,m=1,\dots,r. \tag{3}$$

Proof. As observed in [9], (i)$\Longrightarrow$(ii) is a direct consequence of the Poisson summation formula:

$$\sum_{k\in\mathbb{Z}^d} f(x-k)=\sum_{k\in\mathbb{Z}^d}\widehat{f}(2\pi k)e^{i2\pi k\cdot x}$$

for every compactly supported distribution f on $\mathbb{R}^d$, where the above series on both sides converge in the sense of distributions. For $\zeta\in\mathbb{C}^d$ and $f(x)=\eta(x)e^{-i\zeta\cdot x}$, since $\widehat{f}(z)=\widehat{\eta}(\zeta+z)$, the Poisson summation formula can be written as

$$\sum_{k\in\mathbb{Z}^d}\eta(x-k)e^{-i\zeta\cdot(x-k)}=\sum_{k\in\mathbb{Z}^d}\widehat{\eta}(\zeta+2\pi k)e^{i2\pi k\cdot x} \tag{4}$$

for every compactly supported distribution η on $\mathbb{R}^d$ and $\zeta\in\mathbb{C}^d$. Suppose that (ii) fails. Then there exist $\zeta\in\mathbb{C}^d$ and $c_1,\dots,c_r\in\mathbb{C}$ such that $|c_1|+\dots+|c_r|\ne0$ and (2) holds. Define $\eta:=\sum_{\ell=1}^r c_\ell\phi_\ell$. Then η is a compactly supported distribution on $\mathbb{R}^d$. By (2), we have $\widehat{\eta}(\zeta+2\pi k)=0$ for all $k\in\mathbb{Z}^d$. Now by the Poisson summation formula in (4), we have

$$e^{-i\zeta\cdot x}\sum_{k\in\mathbb{Z}^d}\sum_{\ell=1}^r c_\ell e^{i\zeta\cdot k}\phi_\ell(x-k)=\sum_{k\in\mathbb{Z}^d}\sum_{\ell=1}^r c_\ell e^{-i\zeta\cdot(x-k)}\phi_\ell(x-k)=\sum_{k\in\mathbb{Z}^d}\eta(x-k)e^{-i\zeta\cdot(x-k)}=0.$$

Since $e^{-i\zeta\cdot x}\ne0$, defining $v_\ell(k):=c_\ell e^{i\zeta\cdot k}$ for all $k\in\mathbb{Z}^d$ and $\ell=1,\dots,r$, we see that (1) holds. This is a contradiction to item (i), since not all $v_1,\dots,v_r$ are identically zero. Thus, we proved (i)$\Longrightarrow$(ii).

(iii)$\Longrightarrow$(i) is trivial. Suppose that (1) holds for some $v_1,\dots,v_r\in l(\mathbb{Z}^d)$. Then it follows trivially from (3) in item (iii) that

$$v_m(n)=\Big\langle \sum_{\ell=1}^r\sum_{k\in\mathbb{Z}^d} v_\ell(k)\phi_\ell(\cdot-k), \tilde\phi_m(\cdot-n)\Big\rangle=0, \qquad \forall\, n\in\mathbb{Z}^d, m=1,\dots,r.$$

Hence, all $v_1,\dots,v_r$ must be identically zero. Therefore, we proved (iii)$\Longrightarrow$(i).

We now prove the key part (ii)$\Longrightarrow$(iii) using induction on r. The claim is obviously true for $r=0$, since the statements are empty. We now prove the claim for $r\ge1$. Define a linear mapping L by

$$L(h)(z):=\sum_{k\in\mathbb{Z}^d}\langle\phi_r(\cdot-k),h\rangle z^k, \qquad z\in(\mathbb{C}\backslash\{0\})^d, h\in\mathscr{D}(\mathbb{R}^d). \tag{5}$$

Since both h and ϕ_r are compactly supported, $L(h)$ is a well-defined Laurent polynomial in the Laurent polynomial ring $\mathbb{C}[z_1,\ldots,z_d,z_1^{-1},\ldots,z_d^{-1}]$. Define

$$D:=\{h\in\mathscr{D}(\mathbb{R}^d)\ :\ \langle\phi_\ell(\cdot-k),h\rangle=0,\quad\forall\, k\in\mathbb{Z}^d,\ell=1,\ldots,r-1\}.$$

We now prove that $L(D)$ is an ideal in $\mathbb{C}[z_1,\ldots,z_d,z_1^{-1},\ldots,z_d^{-1}]$. Clearly, D is a linear subspace of $\mathscr{D}(\mathbb{R}^d)$ and $h(\cdot-k)\in D$ whenever $h\in D$ and $k\in\mathbb{Z}^d$. Consequently, $p+q\in L(D)$ whenever $p,q\in L(D)$. Let $h\in D$ and $p(z)=\sum_{n\in\mathbb{Z}^d}p_nz^n\in\mathbb{C}[z_1,\ldots,z_d,z_1^{-1},\ldots,z_d^{-1}]$ be a Laurent polynomial. Define $g:=\sum_{n\in\mathbb{Z}^d}\overline{p_n}h(\cdot-n)$. Since $\{p_n\}_{n\in\mathbb{Z}^d}$ is a finitely supported sequence on $\mathbb{Z}^d$ and h is compactly supported, we see that g is a well-defined function in $\mathscr{D}(\mathbb{R}^d)$ and consequently, $g\in D$ by $h(\cdot-k)\in D$ for all $k\in\mathbb{Z}^d$. By calculation, we have

$$\begin{aligned}L(g)(z)&=\sum_{k\in\mathbb{Z}^d}\langle\phi_r(\cdot-k),g\rangle z^k=\sum_{n\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}p_n\langle\phi_r(\cdot-k),h(\cdot-n)\rangle z^k\\&=\sum_{n\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}p_n\langle\phi_r(\cdot-k+n),h\rangle z^{k-n}z^n=\sum_{n\in\mathbb{Z}^d}p_nz^nL(h)(z)=p(z)L(h)(z).\end{aligned}$$

Since $g\in D$, this proves $pL(h)=L(g)\in L(D)$. Hence, $L(D)$ is an ideal in $\mathbb{C}[z_1,\ldots,z_d,z_1^{-1},\ldots,z_d^{-1}]$. Define $\mathcal{I}:=L(D)\cap\mathbb{C}[z_1,\ldots,z_d]$, where $\mathbb{C}[z_1,\ldots,z_d]$ is the polynomial ring in d-variables. Then it is trivial to see that $\mathcal{I}$ is an ideal in $\mathbb{C}[z_1,\ldots,z_d]$. Define $V(\mathcal{I}):=\{z\in\mathbb{C}^d\ :\ p(z)=0\quad\forall\,p\in\mathcal{I}\}$. Using proof by contradiction, we now prove that $V(\mathcal{I})\cap(\mathbb{C}\backslash\{0\})^d=\emptyset$. Suppose $V(\mathcal{I})\cap(\mathbb{C}\backslash\{0\})^d\neq\emptyset$. Then there exists $\zeta=(\zeta_1,\ldots,\zeta_d)^\mathsf{T}\in\mathbb{C}^d$ such that $e^{i\zeta}:=(e^{i\zeta_1},\ldots,e^{i\zeta_d})^\mathsf{T}\in(\mathbb{C}\backslash\{0\})^d$ and $p(e^{i\zeta})=0$ for all $p\in\mathcal{I}$. By the definition of the mapping L, for $n\in\mathbb{Z}^d$ and $h\in\mathscr{D}(\mathbb{R}^d)$, we have

$$L(h(\cdot-n))=\sum_{k\in\mathbb{Z}^d}\langle\phi_r(\cdot-k),h(\cdot-n)\rangle z^k=\sum_{k\in\mathbb{Z}^d}\langle\phi_r(\cdot-k+n),h\rangle z^{k-n}z^n=z^nL(h)(z).\tag{6}$$

Consequently, for every $h\in D$, since $L(h)$ is a Laurent polynomial, there exists $n\in\mathbb{Z}^d$ such that $L(h(\cdot-n))(z)=z^nL(h)(z)\in\mathcal{I}$. Therefore, for every $h\in D$, we must have $e^{i\zeta\cdot n}L(h)(e^{i\zeta})=L(h(\cdot-n))(e^{i\zeta})=0$ by $L(h(\cdot-n))\in\mathcal{I}$. Since $e^{i\zeta\cdot n}\neq0$, we conclude that $L(h)(e^{i\zeta})=0$ for all $h\in D$, which, as we shall demonstrate later, leads to a contradiction to our assumption in item (ii).

On the other hand, by induction hypothesis, there exist $\tilde{h}_1,\ldots,\tilde{h}_{r-1}\in\mathscr{D}(\mathbb{R}^d)$ such that

$$\langle\tilde{h}_m,\phi_\ell(\cdot-k)\rangle=\delta(m-\ell)\delta(k),\qquad\forall\,k\in\mathbb{Z}^d\quad\text{and}\quad\ell,m=1,\ldots,r-1.\tag{7}$$

For $h\in\mathscr{D}(\mathbb{R}^d)$, we define

$$Ph:=h-\sum_{\ell=1}^{r-1}\sum_{n\in\mathbb{Z}^d}\langle h,\phi_\ell(\cdot-n)\rangle\tilde{h}_\ell(\cdot-n).$$

By (7), it is trivial to directly check that $Ph\in D$ and

$$\begin{aligned}L(Ph)(z)&=\sum_{k\in\mathbb{Z}^d}\langle\phi_r(\cdot-k),h\rangle z^k-\sum_{\ell=1}^{r-1}\sum_{n\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\overline{\langle h,\phi_\ell(\cdot-n)\rangle}\langle\phi_r(\cdot-k),\tilde{h}_\ell(\cdot-n)\rangle z^k\\&=\sum_{k\in\mathbb{Z}^d}\langle\phi_r(\cdot-k),h\rangle z^k-\sum_{\ell=1}^{r-1}\sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}\langle\phi_\ell(\cdot-k),h\rangle\langle\phi_r(\cdot-n),\tilde{h}_\ell(\cdot-k)\rangle z^n\\&=\sum_{k\in\mathbb{Z}^d}\langle\phi_r(\cdot-k),h\rangle z^k-\sum_{\ell=1}^{r-1}\sum_{k\in\mathbb{Z}^d}\langle\phi_\ell(\cdot-k),h\rangle z^k\Big(\sum_{n\in\mathbb{Z}^d}\langle\phi_r(\cdot-n),\tilde{h}_\ell(\cdot-k)\rangle z^{n-k}\Big).\end{aligned}$$

Setting $z=e^{i\zeta}$ in the above identity and defining $c_r:=1$ and $c_\ell:=-\sum_{n\in\mathbb{Z}^d}\langle\phi_r(\cdot-n),\tilde{h}_\ell\rangle e^{i\zeta\cdot n}=-L(\tilde{h}_\ell)(e^{i\zeta})\in\mathbb{C}$ for $\ell=1,\ldots,r-1$, we conclude that

$$\Big\langle\sum_{\ell=1}^{r}\sum_{k\in\mathbb{Z}^d}c_\ell e^{i\zeta\cdot k}\phi_\ell(\cdot-k),h\Big\rangle=L(Ph)(e^{i\zeta})=0\qquad\forall\,h\in\mathscr{D}(\mathbb{R}^d),$$

where we used the facts that $Ph \in D$ and $L(g)(e^{i\zeta}) = 0$ for all $g \in D$. That is, setting $\eta := \sum_{\ell=1}^r c_\ell \phi_\ell$, we proved

$$\sum_{k\in\mathbb{Z}^d} \eta(x-k)e^{-i\zeta\cdot(x-k)} = e^{-i\zeta\cdot x}\sum_{\ell=1}^r \sum_{k\in\mathbb{Z}^d} c_\ell e^{i\zeta\cdot k}\phi_\ell(\cdot - k) = 0.$$

By the Poisson summation formula in (4), we deduce from the above identity that $\sum_{\ell=1}^r c_\ell \widehat{\phi_\ell}(\zeta+2\pi k) = \widehat{\eta}(\zeta + 2\pi k) = 0$ for all $k \in \mathbb{Z}^d$. Since $c_r = 1 \ne 0$, this is a contradiction to item (ii). This proves $V(\mathcal{I}) \cap (\mathbb{C}\backslash\{0\})^d = \emptyset$. Consequently, the polynomial $z_1 \cdots z_d$ must vanish at all points in $V(\mathcal{I})$. By Hilbert Nullstellensatz, there exists a positive integer m such that $z_1^m \cdots z_d^m \in \mathcal{I}$. Hence, there exists $h \in D$ such that $L(h) = z_1^m \cdots z_d^m$. Set $\tilde{\phi}_r := h(\cdot + (m, \dots, m)^\mathsf{T})$. Hence, by (6), we have

$$L(\tilde{\phi}_r) = L(h(\cdot + (m,\dots,m)^\mathsf{T}))(z) = L(h)(z)z_1^{-m}\cdots z_d^{-m} = 1,$$

which is equivalent to $\langle \tilde{\phi}_r, \phi_r(\cdot - k)\rangle = \delta(k)$ for all $k \in \mathbb{Z}^d$. Since $\tilde{\phi}_r \in D$, this implies that $\langle \tilde{\phi}_r, \phi_\ell(\cdot - k)\rangle = 0 = \delta(\ell - r)\delta(k)$ for all $k \in \mathbb{Z}^d$ and $\ell = 1, \dots, r-1$. Now define

$$\tilde{\phi}_\ell := \tilde{h}_\ell - \sum_{n\in\mathbb{Z}^d} \langle \tilde{h}_\ell, \phi_r(\cdot - n)\rangle \tilde{\phi}_r(\cdot - n), \qquad \ell = 1, \dots, r-1.$$

It is trivial to deduce from (7) and the above identities that (3) is satisfied. Since $\tilde{\phi}_1, \dots, \tilde{\phi}_r \in \mathscr{D}(\mathbb{R}^d)$, the claim holds for r. Now by induction, we see that (ii)$\Longrightarrow$(iii). ☐

The equivalence between (i) and (ii) of Main Theorem has been established in [7] (and in [9] for $r = 1$). The existence of compactly supported dual functionals in (iii) is a special case of [1, Theorem 1.3] and [11]. The operator $L(h)$ in (5) is linked to the bracket product. For the bracket product and its applications in approximation theory and wavelet analysis, for example, see [8, 10] and references therein.

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